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ANHARMONIC BETATRON MOTION IN FREE ELECTRON LASERS

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In this article the trajectory of an electron in an FEL is calculated, including nonlinear, off-axis focusing terms in the transverse equations of motion, and the emittance growth due to the nonlinear terms is estimated.

1. Introduction

A clear understanding of the motion of an electron as it passes through the magnetic field of a wiggler of a free electron laser with flat or curved magnetic pole pieces has already proven to be useful in designing magnets which can focus the electron beam and minimize degradation of the beam emittance, preventing a possible degradation of the performance of the FEL [1]. We reexamine this issue, including in the expression for the magnetic field, terms which are higher order in the displacement of the particle from the axis of the wiggler and which have thus far been neglected. The motivation for undertaking this analysis is to examine whether or not these terms, although small, can have a secular effect on such parameters as beam position or emittance over the length of the wiggler. In light of the desire to use very long wigglers in current designs for high-power FELs, it is useful to understand such secular effects.

Previous related work includes ref. [4], where an estimate is made of the parallel-velocity perturbation from the nonlinear focusing terms. In ref. [5] an analysis of the electron orbits, including nonlinear terms, was obtained for the case of a wiggler which had a magnetic field which did not vary in the x-direction, corresponding to wide, flat magnetic pole pieces. In ref. [6] the onset of chaotic orbits is studied in helical wigglers, with space-charge and an axial guide field included.

In ref. [1] the magnetic field B to zeroth order varies sinusoidally with z and is oriented in the y-direction. This zeroth-order field produces the familiar wiggle motion of an electron parallel to the x–z plane. Maxwells equations require the existence of x- and z-components of the magnetic field, although they can be made to vanish along the axis of the wiggler. The field will have a finite field gradient, however, and the interaction of the fields with the parallel and wiggle motion of a finite-sized beam (in particular finite y) will produce a restoring force in the y-direction, linear in y, yielding harmonic motion (see e.g. ref. [1]). This is known as betatron motion and occurs on a longer length scale λ_β than the wiggle motion [$\lambda_\beta \approx (\gamma/a_w)\lambda_w$, where γ is the Lorentz factor of the beam, a_w the dimensionless vector potential and $\lambda_w = 2\pi/k_w$ the wiggle period]. This provides a natural focusing mechanism for the electron beam in the y-direction but does nothing to prevent free expansion of the beam in the x-direction. Use of an external quadrupole field can provide the required focusing in the x-direction but at the expense of an increased emittance [1]. Alternatively, curved pole faces create a gradient of B_y in the x-direction which also produce the required betatron motion in this direction [1]. However, in this case (in contrast to the case of quadrupole focusing) the x and y motion is such as to maintain a constant velocity perpendicular to z (i.e. constant $v_\perp$). This helps to minimize degradation of the optical beam since the particles maintain a nearly constant parallel velocity.

As will be shown, when the next-higher-order terms in the magnetic field are kept, the equations of motion in the x- and y-directions become coupled. When $k_w x$ and $k_w y$ are small (as they are for most FELs), this coupling is small and the equations are like those of weakly coupled harmonic oscillators. The small coupling means that at any particular z the motion is roughly that of a harmonic oscillator without coupling. The effect of the coupling is to transfer "energy" from one oscillator to another. The amplitude of the betatron x motion is slowly exchanged with the amplitude of the betatron y motion, with the amplitudes oscillating back and forth. The projection of the particle motion onto the x–y plane is thus an ellipse whose elements change on a relatively long length scale. Thus a third length scale is introduced into the motion of a particle, the "precession length" λ_p [of order $\gamma\lambda_w/(a_w k_w^2 r^2)$, where r is the displacement of a particle from the wiggler axis]. The word "precession" is set in quotation marks since all of the orbital elements change slowly, not just the periapsis as in the discussion of planetary orbits. The effect of this precession is to increase the amount of x, y, x', y' phase space that a particle will sample in its transit of the wiggler. The betatron frequency is

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also slightly shifted and is of magnitude $2\pi/\lambda_p$. Note that the shift depends on the off-axis displacement.

The outline of this paper is as follows. In section 2, we obtain an expression for the magnetic field of the wiggler which is a Taylor expansion in powers of x and y. In section 3 we write down the Lorentz-force equation expressed as a function of z rather than t. In section 4 we perform a two-length-scale analysis which yields an expression for rapid wiggle motion and two second-order coupled differential equations for the slowly varying "guiding-center" equations. In section 5, a paraxial approximation is used to obtain a set of four first-order differential equations that eliminate the betatron motion from the guiding-center equations. These equations express the position of the particle in terms of the amplitude and phase of the betatron x and y motion. In section 6 we obtain an analytic solution to the amplitude equations for a particular choice of the magnetic-field constants. In section 7 we make an estimate of the emittance growth due to the presence of the nonlinear terms.

2. Specification of the magnetic field

We make the following five assumptions about the wiggler magnetic field $\boldsymbol{B} = mc^2\boldsymbol{b}/e$:

(1) the field is well approximated by a vacuum field:

$$\nabla \times \boldsymbol{b} = \nabla \cdot \boldsymbol{b} = 0,$$

so that

$$\boldsymbol{b} = -\nabla\chi \quad \text{and} \quad \nabla^2\chi = 0, \tag{1}$$

where χ is the magnetic potential;

(2) Along the wiggler axis ($x = y = 0$) the field is that of a perfect wiggler (with no harmonics of the wiggler wave number k_w present):

$$\begin{aligned} &\frac{\partial\chi}{\partial y}(x=0,\ y=0) = -b_0 \cos k_w z, \\ &\frac{\partial\chi}{\partial x}(x=0,\ y=0) = \frac{\partial\chi}{\partial z}(x=0,\ y=0) = 0; \end{aligned} \tag{2}$$

(3) reflection symmetry about x = 0;

$$\chi(-x,\ y) = \chi(x,\ y); \tag{3}$$

(4) reflection antisymmetry about y = 0:

$$\chi(x,\ -y) = -\chi(x,\ y); \tag{4}$$

(5) χ may be expanded as a Taylor series about x = y = 0:

$$\chi = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} M_{mn} x^m y^n \cos k_w z. \tag{5}$$

Assumptions (2) and (5) immediately lead to:

$$M_{01} = -b_0; \quad M_{10} = 0; \quad M_{00} = 0. \tag{6}$$

Use of assumptions (1) and (5) leads to the recursion relation:

$$(m+1)(m+2)M_{m+2,n} + (n+1)(n+2)M_{m,n+2} - k_w^2 M_{m,n} = 0. \tag{7}$$

The symmetry conditions (3) and (4) require that

$$M_{m,n} = 0 \text{ if } m \text{ is odd or } n \text{ is even}. \tag{8}$$

We may thus write down the first six terms of the Taylor series:

$$\chi = \frac{-b_0}{k_w} \cos Z \left[Y + dX^2Y + \left(\frac{1}{6} - \frac{d}{3}\right)Y^3 + cX^4Y + \left(\frac{d}{6} - 2c\right)X^2Y^3 + \frac{1}{20}\left(\frac{1}{6} - \frac{2}{3}d + 4c\right)Y^5 \right]. \tag{9}$$

Here $X \equiv k_w x$, $Y \equiv k_w y$ and $Z \equiv k_w z$, while c and d are arbitrary constants. If X and Y are considered first-order quantities, eq. (9) is correct through order 5. Even orders in the potential are not present. The recursion relation [eq. (7)] dictates that for each additional higher odd order one new arbitrary constant is introduced.

The potential used in ref. [1],

$$\chi = -\frac{b_0}{k_y} \cosh k_x x \sinh k_y y \cos k_w z, \quad k_x^2 + k_y^2 = k_w^2, \tag{10}$$

is a special case of eq. (9). If we choose $d = (k_x/k_w)^2/2$ and $c = (k_x/k_w)^4/24$, then eq. (9) is the fifth-order expansion of eq. (10). For equal harmonic focusing in the x- and y-directions, $k_x = k_y = k_w/\sqrt{2}$ so that $d = 1/4$ and $c = 1/96$. Eq. (9), however, allows an arbitrary c even for equal harmonic focusing in the two directions. For fields which are independent of x, $d = c = 0$.

3. The equations of motion

We assume that the transverse motion of an electron as it passes through the wiggler is dominated by the interaction with the wiggler magnetic field only. The interaction with the fields of the other electrons and with the growing electromagnetic wave, although crucial in changing the parallel velocity and total energy, is ignored in this calculation of the transverse position and velocity. Under this approximation the transverse force on the particle is simply proportional to the transverse component of $\boldsymbol{v} \times \boldsymbol{b}$, where $\boldsymbol{v}$ is the particle velocity. When the independent variable is Z rather than t (time) the equations of motion become

$$\begin{aligned} X'' &= \frac{W_0^2}{u}\left[h_y - Y'h_z - X'(Y'h_x - X'h_y)\right], \\ Y'' &= \frac{W_0^2}{u}\left[X'h_z - h_x - Y'(Y'h_x - X'h_y)\right], \\ u' &= W_0^2(Y'h_x - X'h_y). \end{aligned} \tag{11}$$

Here an apostrophe indicates a derivative with respect to Z, $u \equiv \dot{Z}/\dot{Z}_0$, where the dot indicates a derivative with respect to time t, $W_0^2 \equiv b_0/(\gamma\beta_0 k_w) \cong \sqrt{2}\, a_w/\gamma$, $c\beta_0 = \dot{Z}_0/k_w$ is the parallel velocity at $Z=0$, and $h \equiv b/b_0$.

4. Guiding-center equations

To obtain better physical insight and more numerical utility, we wish to extract the wiggle motion analytically from eq. (11). As in ref. [1], we divide the problem into short (or "rapid") and long ("slow") length scales. Let $X = X_0 + X_1$ and $Y = Y_0 + Y_1$, where X_1, Y_1 vary on scales of order 2π and X_0, Y_0 vary on scales greater than or of order of $2\pi/(d^{1/2} W_0^2)$. We shall assume that X_0 and Y_0 are first-order quantities and that W_0^2 is a second-order quantity satisfying

$$1 \gg X_0,\ Y_0,\ W_0 \gg X_0^2,\ Y_0^2,\ W_0^2. \tag{12}$$

We also represent the field h as a sum of short- and long-length scale quantities as a result of its dependence on X and Y. We treat products of short-length-scale quantities that have an odd number of factors as being a short-length-scale quantity, whereas an even number of factors results (when averaged over a wiggler wavelength) in a long-length-scale, slowly varying quantity.

We find that through fourth order, integration of the fast component of eq. (11) yields

$$\begin{aligned} X_1 &= -\left(1 + dX_0^2 + \left(\tfrac{1}{2} - d\right) Y_0^2\right) W_0^2 \cos Z, \\ Y_1 &= 2dX_0Y_0W_0^2 \cos Z. \end{aligned} \tag{13}$$

Above, the slow quantities were treated as constant during the integration, and higher harmonics in Z were ignored. The quantity u_1 equals zero through fourth order.

When these solutions for X_1 and Y_1 are substituted into eq. (11), and when averages over the short length scale are taken, the result is two coupled second-order differential equations for X_0 and Y_0 (through seventh order):

$$\begin{aligned} X_0'' &= -AW_0^4 X_0\left(1 + CX_0^2 + DY_0^2\right), \\ Y_0'' &= -BW_0^4 Y_0\left(1 + EX_0^2 + FY_0^2\right). \end{aligned} \tag{14}$$

Here

$$\begin{aligned} A &= d; \\ B &= \tfrac{1}{2} - d; \\ C &\equiv (d^2 + 2c)/A; \\ D &\equiv (d + d^2 - 6c)/A; \\ E &\equiv (d + d^2 - 6c)/B; \\ F &\equiv \left(2c + d^2 - 4d/3 + \tfrac{1}{3}\right)/B. \end{aligned} \tag{15}$$

Multiplying the top of eq. (14) by BEX_0' and the bottom of eq. (14) by ADY_0' and adding the results yields the "energy" equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z}\Big[& BEX_0'^2 + ADY_0'^2 + ABW_0^4 \\ & \times \left(EX_0^2 + DY_0^2 + \frac{CE}{2} X_0^4 + \frac{DF}{2} Y_0^4 + DEX_0^2Y_0^2 \right)\Big] \\ &= 0. \end{aligned} \tag{16}$$

In FELS with smaller values of γ it can be a better approximation to make W_0^2 a first-order quantity in which the following ordering is valid:

$$1 \gg X_0,\ Y_0,\ W_0^2 \gg X_0^2,\ Y_0^2,\ W_0^4. \tag{17}$$

Using the ordering in eq. (17) we find that Y_1 is unaltered but that X_1 has an additional term:

$$\begin{aligned} X_1 = &-\left[1 + dX_0^2 + \left(\tfrac{1}{2} - d\right) Y_0^2 + \left(\tfrac{1}{8} + 3d/4\right) W_0^4\right] \\ &\times W_0^2 \cos Z. \end{aligned} \tag{18}$$

We find that eqs. (14) and (15) are then modified only in the sense that A and B must be redefined:

$$\begin{aligned} A &= d + \left(\frac{5d}{8} + \frac{3d^2}{4} + \frac{3c}{2}\right) W_0^4; \\ B &= \tfrac{1}{2} - d + \left(\frac{3}{16} + \frac{d}{8} - \frac{3d^2}{4} - \frac{3c}{2}\right) W_0^4. \end{aligned} \tag{19}$$

(Eqs. (14) had been independently derived previously in ref. [4] for the case of the potential of ref. [1] and the

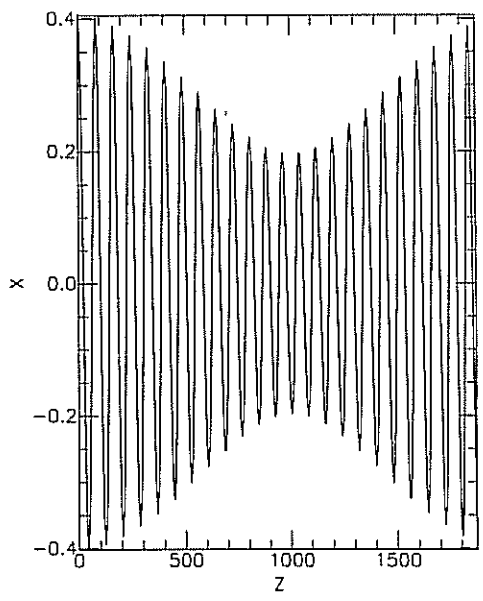

Fig. 1. The X-guiding-center position of a particle when anharmonic terms are included in the equations of motion. (The parameters are listed in table 1). The particle's betatron amplitude is no longer constant; rather, the X and Y amplitudes are exchanged in a precession length scale $= 2\pi/K_p$.

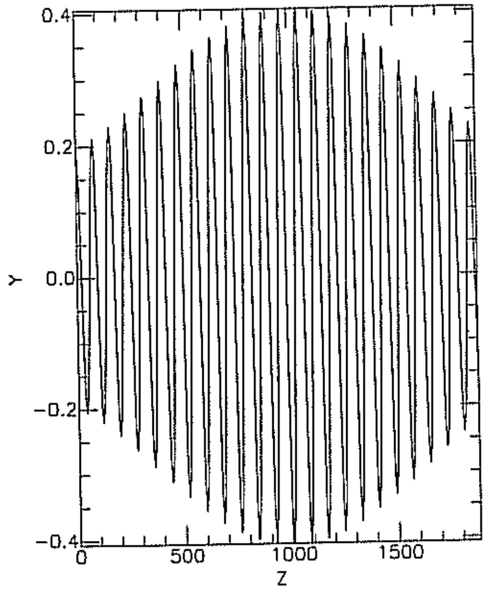

Fig. 2. The Y guiding center. As the X-amplitude decreases, the Y-amplitude increases.

limit when W_0^2 is a second-order quantity. For the case $c = d = 0$ (flat pole pieces, with the fields independent of x) eqs. (14) reduce to eq. (4.11) of ref. [5] when terms to the same order are compared).

In figs. 1 and 2 we numerically integrate eqs. (14), yielding the guiding-center positions X_0 and Y_0 as functions of Z. The parameters were chosen so as to illustrate the precession of the betatron orbits. The amplitudes are interchanged over a distance (for the case shown) of about 14 betatron wavelengths.

It is interesting to raise the question [3] as to whether or not the coupling between X_0 and Y_0 can be minimized by a judicious choice of constants d and c. In fact, if $c = (d + d^2)/6$, the coupling constants D and E both vanish. The amplitudes then behave as independent harmonic oscillators with small cubic nonlinearities. A carefully constructed magnetic pole face (where the height and curvature varied as functions of X) could produce the required c and d. The cubic nonlinearity produces an amplitude-dependent betatron frequency, however, which produces emittance growth quite similarly to the case for which the amplitudes are coupled. Our subsequent analysis assumes a more generic D and E which do not, in general, vanish.

5. Equations for the betatron amplitude and phase

The differential equations in eq. (14) are in the form of harmonic oscillators with weak, nonlinear coupling coefficients. Since at any point in Z the basic motion is harmonic, it is natural to try a substitution in which the betatron motion is explicitly eliminated, solving instead for the betatron amplitude and phase. Let

$$X_0 \equiv X_\beta \cos\left(A^{1/2} W_0^2 Z + \phi_x\right),$$
$$Y_0 \equiv Y_\beta \cos\left(B^{1/2} W_0^2 Z + \phi_y\right). \tag{20}$$

Substituting eq. (20) into eq. (14) we obtain for the upper equation terms proportional to $\cos(A^{1/2} W_0^2 Z + \phi_x)$ and $\sin(A^{1/2} W_0^2 Z + \phi_x)$ with analogous terms for the lower equation. We make a WKB-like approximation in that we eliminate terms proportional to X_β'', ϕ_x'', Y_β'' and ϕ_y''. Further we drop terms proportional to the cosine or sine of twice the betatron frequency, as they average to zero over a betatron wavelength. The result is a set of four first-order coupled equations for the betatron amplitude and phase:

$$\begin{aligned}
X_\beta' &= -\frac{A^{1/2} D W_0^2}{8} X_\beta Y_\beta^2 \sin 2\Delta\phi, \\
Y_\beta' &= \frac{B^{1/2} E W_0^2}{8} Y_\beta X_\beta^2 \sin 2\Delta\phi, \\
\phi_x' &= \frac{A^{1/2} W_0^2}{8}\left[3CX_\beta^2 + (2 + \cos 2\Delta\phi) D Y_\beta^2\right], \\
\phi_y' &= \frac{B^{1/2} W_0^2}{8}\left[(2 + \cos 2\Delta\phi) E X_\beta^2 + 3F Y_\beta^2\right],
\end{aligned} \tag{21}$$

where $\Delta\phi \equiv (A^{1/2} - B^{1/2}) W_0^2 Z + \phi_y - \phi_x$.

6. Analytic solution for a special case

For the magnetic potential treated in ref. [1], for the case of equal harmonic focusing in the X- and Y-directions, and when W_0^2 is regarded as a second-order quantity, the constants c, d and A–F take on particularly simple values:

$$d = \tfrac{1}{4},\ c = \tfrac{1}{96},$$
$$A = B = \tfrac{1}{4},$$
$$D = E = 1,$$
$$C = F = \tfrac{1}{3}.$$

Eqs. (21) then take on the simpler form

$$\frac{\mathrm{d}}{\mathrm{d}z} U = -\mu U V \sin 2\Delta\phi,$$
$$\frac{\mathrm{d}}{\mathrm{d}z} V = \mu U V \sin 2\Delta\phi,$$
$$\frac{\mathrm{d}}{\mathrm{d}z} \Delta\phi = -\frac{\mu}{2}(1 + \cos 2\Delta\phi)(V - U). \tag{22}$$

Here $U \equiv X_\beta^2$, $V \equiv Y_\beta^2$ and $\mu = W_0^2/8$. For this case it is seen that the sum of the squares of the betatron amplitudes is constant, i.e.,

$$U + V = \text{constant} \equiv C_1. \tag{23}$$

This corresponds to a constant energy as given by eq. (16). By expressing eq. (22) in terms of $U + V$ and

Table 1
Wiggler parameters

	ELF [a,b]	IMP [a]	PALADIN [a]	EXAMPLE [b]
r [cm]	0.6	0.4	0.5	0.5
γ	6.9	20	100	30
$W_0^2=\sqrt{2}\,a_w/\gamma$	0.52	0.20	0.017	0.15
L [cm]	400	50	2500	2400
k_w [cm^{-1}]	0.64	0.63	0.78	0.78
k_β [cm^{-1}]	0.15	0.062	0.0065	0.059
k_p [cm^{-1}]	5.1×10^{-3}	9.5×10^{-4}	2.5×10^{-4}	3.4×10^{-3}
$N_w=k_wL/2\pi$	40	55	302	300
$N_\beta=k_\beta L/2\pi$	9.5	5.4	2.6	22
$N_p=k_pL/2\pi$	0.32	0.083	0.10	0.91

[a] ELF (Electron Laser Facility), PALADIN and IMP (Intense Microwave Prototype) are past, present and future FEL experiments at the Lawrence Livermore National Laboratory.

[b] Note that ELF used quadrupole focusing (not curved pole pieces), and that EXAMPLE indicates the parameters used in figs. 1–5 that were chosen to illustrate the effects of the nonlinear terms.

$U-V$ it is easy to obtain a second constant of the motion:

$$UV(1+\cos 2\Delta\phi)=\text{constant}\equiv C_2. \tag{24}$$

After one additional integration and considerable algebra we obtain the following exact solution to eq. (22):

$$X_\beta^2=\frac{C_1}{2}\left\{1\pm\left[\frac{\alpha}{2}\left(1-\sin(2K_pZ+\theta_0)\right)\right]^{1/2}\right\},$$

$$Y_\beta^2=\frac{C_1}{2}\left\{1\mp\left[\frac{\alpha}{2}\left(1-\sin(2K_pZ+\theta_0)\right)\right]^{1/2}\right\},$$

$$\Delta\phi=\frac{1}{2}\cos^{-1}\left(\frac{2-3\alpha-\alpha\sin(2K_pZ+\theta_0)}{2-\alpha+\alpha\sin(2K_pZ+\theta_0)}\right). \tag{25}$$

Here,

$$\alpha=1-\frac{2C_2}{C_1^2},\quad K_p=\sqrt{2C_2}\,\frac{W_0^2}{8},\quad \Delta\phi_0=\Delta\phi(z=0),$$

$$\theta_0=\sin^{-1}\left(-1+\frac{4C_2}{\alpha C_1^2}\frac{(1-\cos 2\Delta\phi_0)}{(1+\cos 2\Delta\phi_0)}\right). \tag{26}$$

In eq. (25) the solution alternates between use of the upper sign and the lower sign, with the transition between signs occurring when $X_\beta=Y_\beta$. The solution is thus periodic in Z with period $2\pi/K_p$. Note that care must be taken to choose the correct branch of the $\cos^{-1}$ in eq. (25) and of the $\sin^{-1}$ in eq. (26). We may use eqs. (21) and (25) to solve for the sum of the betatron phases $\phi_+\equiv\phi_y+\phi_x$:

$$\phi_+=\frac{W_0^2}{8}C_1Z+\phi_{+0}+\tan^{-1}[g(\theta)]-\tan^{-1}[g(\theta_0)]$$
$$+\pi\,\mathrm{Int}\left(1+\frac{\theta-\pi}{2\pi}\right). \tag{27}$$

Here,

$$g(\theta)=\frac{(2-\alpha)\tan(\theta/2)+\alpha}{2(1-\alpha)^{1/2}},$$

$$\theta=2K_pZ+\theta_0,$$

$$g(\theta_0)=g[\theta(z=0)].$$

The final term in eq. (27) ensures continuity in ϕ_+ when the principal value of the $\tan^{-1}$ is taken. From eq. (27) the x- and y- betatron phases are obtained individually via $\phi_x=(\phi_+-\Delta\phi)/2$ and $\phi_y=(\phi_++\Delta\phi)/2$.

The analytic solution is of interest because it provides the scaling for the betatron "precession" rate K_p and the change in the betatron wave number, ΔK_β. That is, eqs. (25) and (26) imply

$$K_p=\frac{1}{4\sqrt{2}}X_\beta Y_\beta(1+\cos 2\Delta\phi)^{1/2}W_0^2,$$

$$K_\beta=\frac{W_0^2}{2}+\Delta K_\beta,$$

where

$$\Delta K_\beta\equiv\frac{1}{2}\left\langle\frac{d\phi_+}{dZ}\right\rangle$$
$$=\left\{X_\beta^2+Y_\beta^2+X_\beta Y_\beta[2(1+\cos 2\Delta\phi)]^{1/2}\right\}\frac{W_0^2}{16}. \tag{28}$$

Here $\langle\ \rangle$ indicates average over a betatron wavelength.

Eq. (28) indicates that the length scale over which the x and y amplitudes of the betatron motion return to their values at $z=0$ is $L_p\equiv 2\pi k_w/K_w\approx\gamma\lambda_w/(a_wk_w^2r^2)$ for a particle at radius r. The number of precession periods, N_p, is L/L_p, where L is the wiggler length. We have tabulated this quantity along with the number of betatron wavelengths N_β and wiggle wavelengths N_w in table 1 for some past, present and future FELs. (Note that ELF is included in the table even though the focusing in ELF was by quadrupoles. The table is meant to give a feel for the relative importance of these effects for various classes of FEL). Although the higher-power and higher-frequency FELs are longer in physical length, their increase in γ also greatly increases their precession length, implying that these anharmonic effects become more negligible at higher power. IMP and Paladin have $N_p \approx 0.08$–0.10, and in an ELF-like FEL nearly one third of a precession period will have occurred for an electron at the outer radius of the beam. This result implies that the precise location of the particles in these FELS will bear little resemblance to the positions predicted by inclusion of the harmonic focusing terms only. However, the effect on the macroscopic beam quantities is less clear.

7. Estimate of the emittance growth from nonlinear focusing

In order to examine the effect on beam quantities, we integrated eq. (21) using an ensemble of particles, loaded such that the density in x, y, x', y' phase space was uniform, using the transverse loading algorithm used in the FEL computer code FRED [2]. Using the beam parameters labeled "example" in table 1, and starting with a beam radius that was initially perfectly matched to the velocity spread, although intentionally off-axis, we computed the evolution of the beam as a function of z. We computed the beam radius R, beam-centroid position X_c, Y_c, and the emittance E which we define such that in the absence of anharmonic terms the linear focusing would result in constant emittance:

$$E_x \equiv \left(\langle \Delta X^2\rangle\langle \Delta X'^2\rangle - \langle \Delta X\, \Delta X'\rangle^2\right)^{1/2},$$
$$E_y \equiv \left(\langle \Delta Y^2\rangle\langle \Delta Y'^2\rangle - \langle \Delta Y\, \Delta Y'\rangle^2\right)^{1/2},$$
$$E \equiv \left(E_x^2 + E_y^2\right)^{1/2}, \tag{29}$$

where

$$\Delta X \equiv X_i - X_c, \quad \Delta Y \equiv Y_i - Y_c,$$
$$\Delta X' \equiv X_i' - X_c', \quad \Delta Y' = Y_i' - Y_c',$$
$$X_c \equiv \langle X_i \rangle, \quad Y_c \equiv \langle Y_i \rangle,$$
$$X_c' \equiv \langle X_i' \rangle, \quad Y_c' \equiv \langle Y_i' \rangle.$$

Here N is the number of simulation particles, i indicates the ith particle and $\langle\ \rangle = (1/N)\Sigma_{i=1}^N$.

Figs. 3, 4 and 5 illustrate the evolution of the centroid, radius and emittance, respectively, of a beam in which the original coordinates of the centroid are the same as the initial coordinates of the particle in figs. 1 and 2. Initially the evolution of the centroid follows the coordinates of the particle. However, since each particle in the beam has a slightly different period of betatron motion due to the nonlinear focusing terms, the beam as a whole loses its coherent betatron motion. The

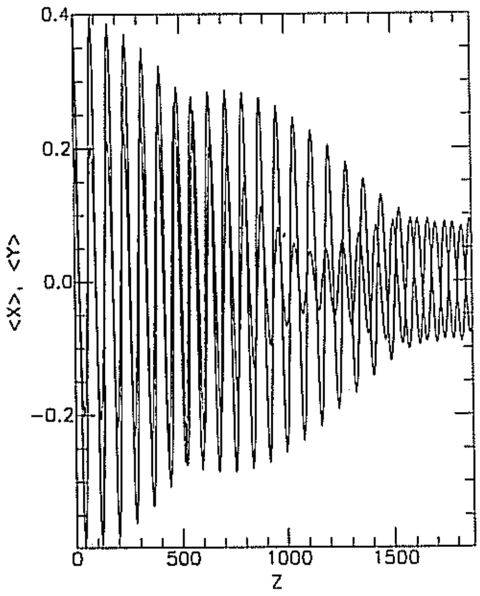

Fig. 3. The X- and Y-positions of the beam centroid as a function of Z. The initial coordinates for the beam centroid are the same as for the individual particle in figs. 1 and 2. The initial radius of the beam is 0.08. At first the centroid undergoes betatron motion, with the centroid X- and Y-amplitudes interchanging as in figs. 1 and 2. As phase mixing occurs, however, the centroids tend to zero.

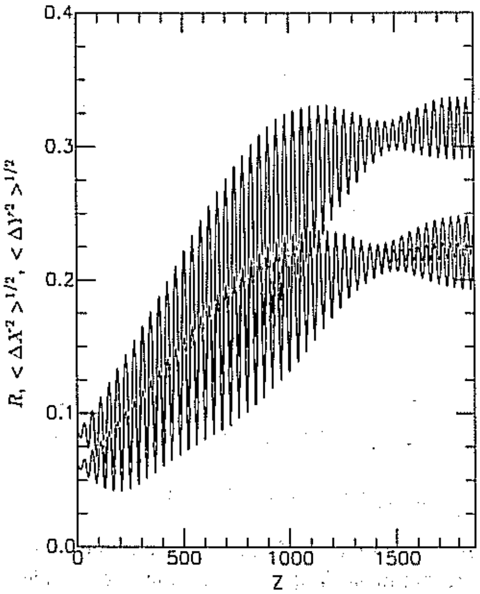

Fig. 4. The beam radius and width in the X- and Y-directions. As the centroid positions tend to zero, the beam broadens, while undergoing betatron oscillations.

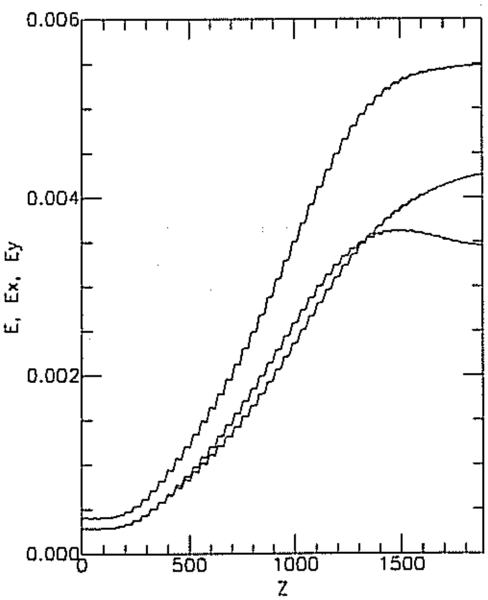

Fig. 5. As phase mixing occurs, the beam initially offset with low emittance is converted to a broader, hotter beam (higher emittance) with no displacement.

centroid of the beam tends to zero as the radius of the beam expands. In this example the beam was initially matched in the sense that it was not undergoing radial oscillations, but displaced from the magnetic axis. The final state is one in which the beam is both matched in radius and is not oscillating about the axis. Both the emittance and radius of the beam have increased as a result.

We may estimate the emittance growth rate from the nonlinear terms based on the preceding discussion. Energy conservation implies

$$H_0 \cong \tfrac{1}{2}\left(X_c^2 + \langle \Delta X^2 \rangle + \frac{X_c'^2}{K_\beta^2} + \frac{\langle \Delta X'^2 \rangle}{K_\beta^2} + Y_c^2 + \langle \Delta Y^2 \rangle + \frac{Y_c'^2}{K_\beta^2} + \frac{\langle \Delta Y'^2 \rangle}{K_\beta^2} \right). \tag{30}$$

H_0 is an approximate constant of the motion (the higher-order terms have been neglected in this expression). Here $K_\beta = W_0^2/2$. For a rotationless beam

$$E = \left(\langle \Delta X^2 \rangle \langle \Delta X'^2 \rangle + \langle \Delta Y^2 \rangle \langle \Delta Y'^2 \rangle \right)^{1/2}.$$

There are two major effects of the nonlinearity:

(1) the betatron frequency becomes amplitude-dependent so that phase mixing results, and

(2) the X and Y amplitudes are coupled and thus energy is exchanged.

Phase mixing and virialization results in a final state such that

$$X_c = Y_c = 0,$$

$$\langle \Delta X^2 \rangle = \frac{\langle \Delta X'^2 \rangle}{K_\beta^2} = \langle \Delta Y^2 \rangle = \frac{\langle \Delta Y'^2 \rangle}{K_\beta^2}.$$

Thus $E_{\text{final}} = \sqrt{2} K_\beta \langle \Delta X^2 \rangle_{\text{final}}$ and $\langle \Delta X^2 \rangle_{\text{final}} = H_0/2$, so that

$$E_{\text{final}} = \frac{1}{\sqrt{2}} K_\beta H_0. \tag{31}$$

The emittance grows due to the phase mixing. The (dimensionless) growth length L_g is defined by

$$\Delta K_\beta L_g \cong \pi. \tag{32}$$

Here ΔK_β is the difference in betatron frequencies between the outermost particle and a central particle. Thus the rate of emittance growth is roughly given by

$$\frac{dE}{dZ} \approx \begin{cases} \frac{\Delta K_\beta}{\pi}(E_{\text{final}} - E_0), & Z < L_g, \\ 0, & Z > L_g. \end{cases} \tag{33}$$

(For $Z > L_g$ the emittance approaches the asymptotic value given by eq. (31). Of course other effects such as space-charge or wiggler errors which have been neglected may lead to continued emittance growth).

Finally, to determine the effect on FEL performance, we incorporated eqs. (14), (15) and (19) into the FEL code FRED and compared runs with and without the anharmonic terms for parameters used to simulate IMP. Note that the assumption of constant a_w and γ is relaxed when the integration in FRED is carried out. The final power was insensitive to the presence or absence of the terms. The simulations showed only a 1% final power difference.

8. Summary and conclusions

We have reanalyzed the transverse equations of motion for an electron transiting a wiggler, including terms of higher order in the small quantities $k_w x$, $k_w y$ and $\sqrt{2} a_w/\gamma$ that had not previously been included. We find that the equations for the guiding-center particle coordinates x and y can be expressed as a pair of weakly coupled harmonic oscillator equations [eq. (14)]. The resulting motion is approximately harmonic betatron motion but with amplitudes and phases that precess on a relatively long length scale. The precessing phase implies an effective correction to the betatron wave number. The length scale increases linearly with γ and with the inverse square of the beam radius, both factors indicating that the effects of the included higher-order terms will diminish for high-power, high-frequency FELs. When ensembles of particles are examined, the nonlinear terms cause the emittance to grow. This is due to the phase mixing caused by the amplitude-dependent betatron frequency. A beam with an initial offset of the beam centroid is thus converted into a beam with a larger radius and higher emittance. Simulations with the FEL code FRED indicate that the fractional change in final power produced is small when the initial beam mismatch is small, however.

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